CREEP ANALYSIS OF ELASTIC BEAMS UNDER CONSTANT TORQUE

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Abstract: *This paper presents a two-dimensional numerical algorithm for creep analysis of the elastic beams under uniform torsion. Torque is assumed to be constant during the whole creep process. Tangential stresses are calculated following the warping function distribution. Material creep behaviour is simulated using the effective stress function. Analysis takes in consideration the torque acting on cross-sectional surface independently on the beam length. The proposed numerical algorithm enables the stress analysis to be carried out regardless of the cross-sectional shapes. Viscoelastic effects of the material are modelled by the creep power law formula. Numerical algorithm was developed in Python code and its effectiveness is validated through the benchmark example.*

Keywords: UNIFORM TORSION, CREEP, EFFECTIVE STRESS FUNCTION

1. Introduction

The viscoelastic, viscoplastic and plastic effects form two main groups of phenomena acting on any material. First phenomenon is creep, which results in creep strains that developed over a finite time. The second phenomenon is plasticity, which results in permanent plastic strains independent of the time. Although different in nature, this two phenomena cannot be treated separately.

Important works covering the area of viscoelastic, viscoplastic creep and plasticity are given in Refs. 1-5.

Uniform torsion theory, also referred to as St. Venant's torsion theory, is well described in literature [6-9]. This torsion theory has been, for a long time, a fundamental theory applied by many researchers in the field of torsion. With the evolution of the beam theories, especially the thin-walled beam theory, it was also necessary to include the warping effects. Therefore, in particular problems, non-uniform torsion theory has to be applied [10, 11]. Some numerical solutions to the uniform torsion problem are presented in Refs. 6, 12 and 13.

This paper presents a numerical algorithm for the creep analysis of beams subjected to constant torque. Warping effects are neglected, which means that uniform torsion theory is assumed. The element stiffness matrix and load vectors are derived using the warping function. The proposed numerical algorithm enables the stress analysis to be carried out regardless of cross-sectional shapes. Creep algorithm is defined by effective-stress-function presented by Kojić and Bathe [1,2]. Viscoelastic effects of the material are modeled by the power creep formula. The numerical algorithm code is written in the Python 2.7.11 programming language [14]. The effectiveness of the numerical algorithm discussed is validated through a benchmark example.

2. Basic considerations

3.1 Beam kinematics

Displacement filed is given as

$$\begin{aligned} u(y,z) &= -y\varphi(z) \\ v(x,z) &= x\varphi(z) \\ w(x,y,z) &= \varphi'(z)\omega(x,y) \end{aligned} \tag{1}$$

where u, v and w are the rigid-body translations of the cross-section in the *x*-, *y*- and *z*-directions, respectively; φ is the angle of twist; ω is the warping function. The non-zero strain components are found from the first order strain-displacement relations as

$$\varepsilon_z = \frac{\partial w}{\partial z} = \varphi''\omega = 0 \quad \varphi' = \text{const.}$$

$$\begin{aligned} \gamma_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \varphi'\left(\frac{\partial \omega}{\partial x} - y\right) \\ \gamma_{zy} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \varphi'\left(\frac{\partial \omega}{\partial y} + x\right) \end{aligned} \tag{2}$$

If the linear relation between stress and strain is given in a form of a Hooke's law, the non-zero stress components are

$$\begin{aligned} \tau_{zx} &= G\gamma_{zx} = G\varphi'\left(\frac{\partial \omega}{\partial x} - y\right) \\ \tau_{zy} &= G\gamma_{zy} = G\varphi'\left(\frac{\partial \omega}{\partial y} + x\right) \end{aligned} \tag{3}$$

With G as the shear modulus, stress resultant acting on each cross-section of the beam can be defined as follows

$$T = G I_{\mathrm{t}} \varphi' \tag{4}$$

where T is the torsional moment or torque, while I_{t} represents St. Venant's torsional constant. If the stress components are expressed in the form of the stress resultant, they follow as

$$\begin{aligned} \tau_{zx} &= \frac{T}{I_{\mathrm{t}}}\left(\frac{\partial \omega}{\partial x} - y\right) \\ \tau_{zy} &= \frac{T}{I_{\mathrm{t}}}\left(\frac{\partial \omega}{\partial y} + x\right) \end{aligned} \tag{5}$$

Since there are only two non-zero stress components, the stress deviator takes its form as

$$\mathbf{S}^{\mathrm{T}} = \left\{ \tau_{zx} \quad \tau_{zy} \right\} \tag{6}$$

Considering the Hooke's law in the elastic area, the strain deviator, for this problem, can be expressed by the stress deviator as

$$\mathbf{e} = \frac{\mathbf{S}}{2G} \tag{7}$$

Partial differential equation for the warping function is obtained from one of the equilibrium equations

$$\nabla^2\omega = \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} = 0 \tag{8}$$

Boundary condition for the partial differential equation (8) is

$$\tau_{zx} n_x + \tau_{zy} n_y = 0 \tag{9}$$

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4.2 Elastic creep (time iteration)

In the next two sections basic creep procedure of the numerical algorithm will be presented. First section will describe numerical algorithm in relation to real time. The elastic creep procedure is based on the Kojić's and Bathe's effective stress function [1]. Analysis takes in consideration the torque acting on cross-sectional surface independently on the beam length. The effective stress function is calculated for every time step and for every point in the domain as

$$f({}^{t+\Delta t}\bar{\sigma}) = a^2\, {}^{t+\Delta t}\bar{\sigma}^2 + b\, {}^{w}\gamma - c^2\, {}^{w}\gamma^2 - d^2 \tag{10}$$

where $\bar{\sigma}$ is the effective stress defined as

$$\bar{\sigma} = \sqrt{\frac{3}{2}\left(\tau_{zx}^2 + \tau_{zy}^2\right)} \tag{11}$$

The Eq. (10) is described in detail in [1]. To solve Eq. (10) further information is needed. This information is acquired experimentally and formed in a creep formulae. General formulation for the creep law is

$$\bar{e}^C = f_1(\bar{\sigma})\, f_2(t)\, f_3(\vartheta) \tag{12}$$

where t is the time and ϑ the temperature. If the creep law is given by the power creep formula, it is represented in the following form

$$\bar{e}^C = k_1 \bar{\sigma}^{k_2} t^{k_3} \tag{13}$$

where k_1, k_2 and k_3 are the creep constants, obtained experimentally. Increment of the effective creep strain is the time derivative of Eq. (12) at weighted time $w = t + \alpha\Delta t$

$$\dot{\bar{e}}^C = \Delta t\; f_1({}^{w}\bar{\sigma})\, \dot{f}_2(w)\, f_3({}^{w}\vartheta) \tag{14}$$

The effective stress is obtained by finding the root of the effective stress function given by the Eq. (10). This procedure is, in the most cases, solved numerically (e.g. bisection method, Newton's method, secant method, etc.).

By obtaining the effective stresses for every point in the domain, the stress deviator at the new time step ${}^{t+\Delta t}\mathbf{S}$ is calculated as

$$ {}^{t+\Delta t}\mathbf{S} = 2G \frac{{}^{t+\Delta t}\mathbf{e}'' - (1-\alpha)\Delta t\, {}^{w}\gamma\, {}^{t}\mathbf{S}}{1 + 2G\alpha\Delta t\, {}^{w}\gamma} \tag{15}$$

5.3 Elastic creep (torque correction)

The purpose of this algorithm is to keep the resultant of the inner forces in equilibrium with the external forces applied on the cross-section (between every two time steps). The first time step at $t = 0$ is simple elastic solution of a uniform torsion problem. In every next time step creep defections are applied in a form of a creep law. The inner torsional moment is calculated as

$$T = \int_A \left[\tau_{zy} x - \tau_{zx} y\right] \mathrm{d}A \tag{16}$$

Following the procedure from the previous section, for a constant torque applied at the cross section, there is a relation

$$ {}^{t+\Delta t}\mathbf{S} \le {}^{t}\,\mathbf{S} \tag{17}$$

and therefore

$$ {}^{t+\Delta t}T \le {}^{t}\,T \tag{18}$$

Since the inner torsional moment has to be constant during the whole creep procedure, it will be manually increased. The torsional moment is increased indirectly by increasing the stresses in all points of a domain. The stresses are increased for a segment of the starting elastic solution at $t = 0$. The segment is determined by the difference between starting torsional moment and the torsional moment in the current iteration, for the *i-th* iteration as

$$\begin{aligned} \left(\tau_{zx}\right)_{\mathrm{d}} &= \int_A \frac{T^0 - T^i}{I_{\mathrm{t}}} \left(\frac{\partial \omega}{\partial x} - y\right) \mathrm{d}A \\ \left(\tau_{zy}\right)_{\mathrm{d}} &= \int_A \frac{T^0 - T^i}{I_{\mathrm{t}}} \left(\frac{\partial \omega}{\partial y} + x\right) \mathrm{d}A \end{aligned} \tag{19}$$

where T^0 is torsional moment at time $t = 0$. This procedure is repeated until the following criteria is satisfied

$$T^0 - T^i = e \tag{20}$$

where e is the tolerance constant.

3. Numerical Example

On the basis of the proposed numerical algorithm, a computer program for the creep stress analysis is developed. Program code is written in Python 2.7.11 programing language. The accuracy of the presented model is illustrated by one example. Stresses calculated by the computer program are in the form of the resultant shear stress τ, which is given as

$$\tau = \sqrt{\tau_{zx}^{\ 2} + \tau_{zy}^{\ 2}} \tag{21}$$

From Eq. (21), one should note that the resultant shear stress τ will always be positive since it only represents the intensity of the stress and not the direction.

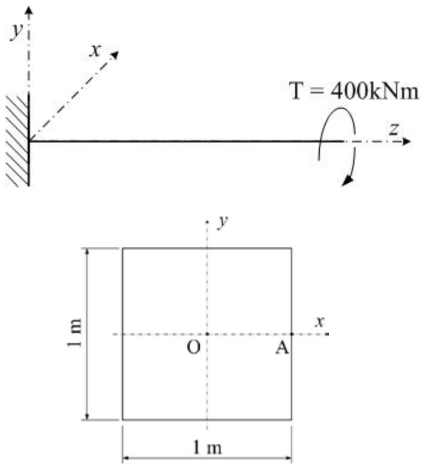

Fig. 1 *Cantilevered square beam*

Figure 1 shows a cantilevered beam of square cross-section. Beam is subjected to a constant torque T = 400kN. Warping effects are neglected. The elastic moduli are E = 200 GPa and G = 100 GPa. The integration parameter α = 1. Creep strains are calculated by the following creep law:

$$\bar{e}^C = 10^{-16} \bar{\sigma}^3 t^2$$

until the steady state creep is reached at time t = 100h. The same example is solved by the Kojić and Bathe in [2]. Figure 2 shows stress distribution over the surface of the cross-section for different time steps. The comparison of the results obtained by this numerical algorithm and the results obtained by Kojić and Bathe is shown in figure 3.

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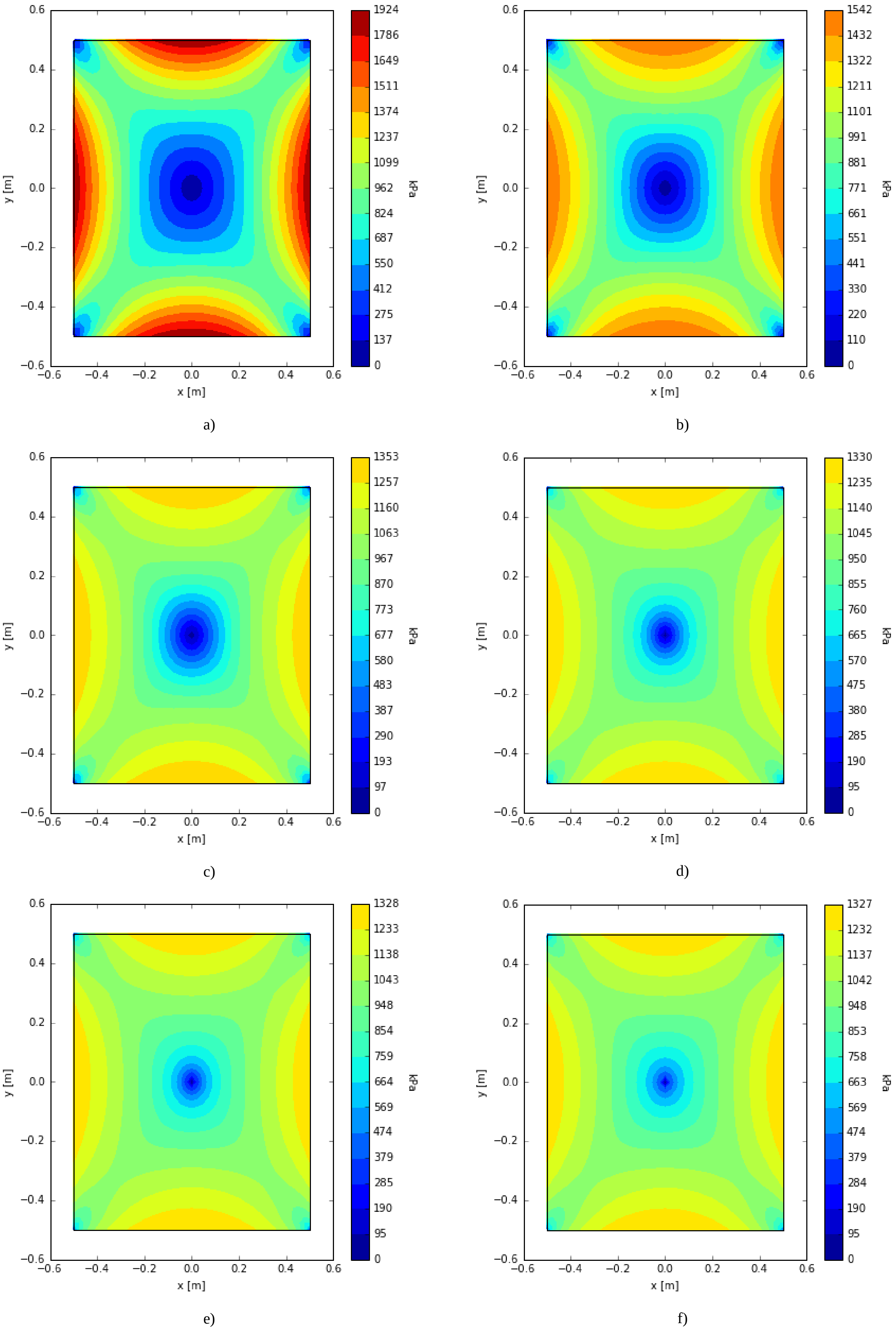

Fig. 2 *Stress distribution over the surface of the cross section for*

a) t = 0, b) t = 20, c) t = 40, d) t = 60, e) t = 80 *and* f) t = 100 *hours*

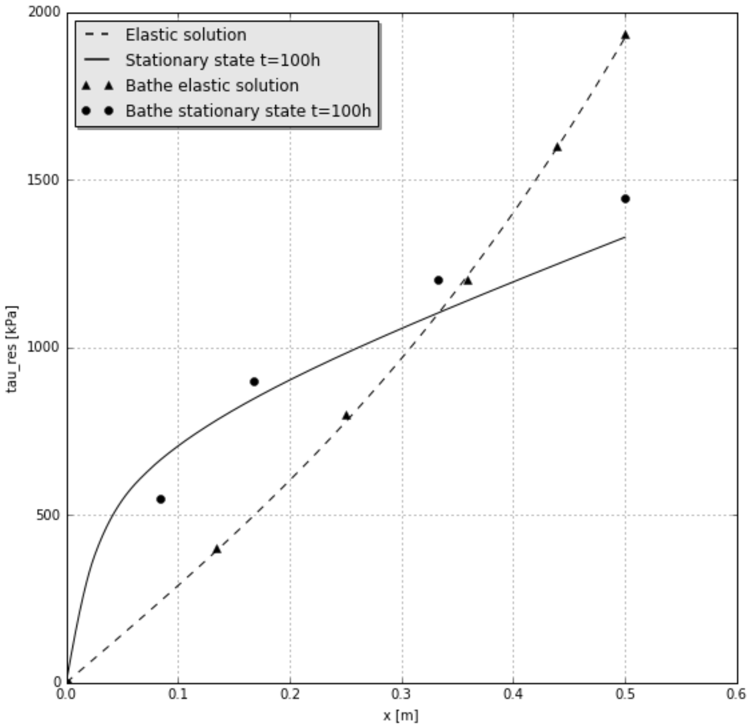

Fig. 3 *Stress distribution over the O – A line*

4. Conclusion

This paper has presented a model for the elastic creep stress analysis of the beams under the constant torque. Numerical algorithm was developed in Python code and its reliability has been verified by the study of one example. The results of the example presented are compared to results of Kojić and Bathe original ones [2]. Further research will extend the proposed algorithm to a model with non-uniform torsion.

5. Acknowledgement

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