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APPLYING A MODIFIED VARIATIONAL ITERATION METHOD TO THE PLUNGE GALLOPING EQUATION

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Abstract: *The plunge galloping is a high-amplitude, low-frequency oscillation of a slender structure, such as iced conductors of a power transmission line or bridge decks, essentially perpendicular to the wind direction. In the paper, an idealized model for the plunge galloping is shortly reviewed and then a slight modification of the variational iteration method, applicable to weakly nonlinear problems, is employed to obtain a system of two amplitude-frequency equations that provide both the transitional and long-term behaviours. The approximate analytical results derived in the paper have been applied to a typical section model and the numerical results are contrasted with those provided by the direct integration of equation of motion.*

Keywords: PLUNGE GALLOPING, VARIATIONAL ITERATION METHOD, NUMERICAL SIMULATION

1. Introduction

Galloping is a wind-induced vibration of a lighting damped slender structure exposed to moderate to high steady winds. It affects mainly the power transmission lines and is characterized by low frequency and large vertical amplitude. The vibration can last for a few hours or several days and causes all sorts of damages, including flashovers between vertically aligned phases, separated insulator strings and even the failure of the hardware.

Different models, with a finite or infinite number of degrees-of-freedom, have been proposed for understanding, controlling and preventing galloping [1-3]. A significant role in this sense has been played by the idealized one-degree-of-freedom model intensively studied in the 1970's concerning translational or rotational galloping. In the paper, this model is shortly reviewed and then the Variational Iteration method is used to derive trustworthy approximate solution for the equation of motion.

2. Formulation of the problem

Consider an elastically suspended cylinder-like body having a constant cross-section of an arbitrarily shape that can perform only a plunge oscillation of magnitude y perpendicular to the direction of a flow of constant velocity $\vec{V}_\infty$. Assuming that the elastic and structural damping forces are linear, the equation of motion is written as

$$m\ddot{y} + r\dot{y} + k\,y = F_y \qquad (1)$$

where m is the body mas on unit length, r and k are the damping and stiffness coefficients and F_y the vertical aerodynamic force. It is given by

$$F_y = C_y(\alpha)\cdot\frac{1}{2}\rho_a\,d\,l\,V_\infty^2 \qquad (2)$$

where ρ_a is air density, l the body length, d a suitable chosen dimension of the cross-section, C_y the vertical force coefficient and α the angle of attack (incidence) of the body to the relative wind (see Fig. 1).

Under the quasi-steady assumption, the coefficient C_y can be expressed as a polynomial in angle α. So, Blevins [4] have used the cubic representation

$$C_y = a_1\alpha + a_2\alpha^2 + a_3\alpha^3 \qquad (3)$$

to describe the force coefficient on a square cross-section. Parkinson and Smith [5], in desiring to provide a more accurate description for the force coefficient of the square section, have proposed an odd polynomial of order seven

$$C_y = a_1\alpha + a_3\alpha^3 + a_5\alpha^5 + a_7\alpha^7 \qquad (4)$$

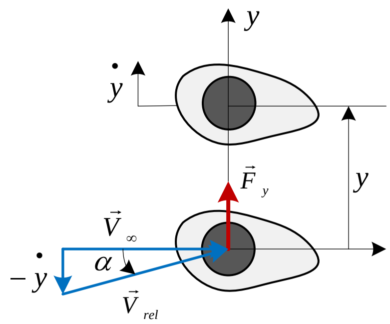

Fig. 1. *Dynamic model for the pure plunge galloping*

In the paper, we have combined the above expressions to yield

$$C_y = a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + a_5\alpha^5 + a_7\alpha^7 \qquad (5)$$

Noting that for small α ones has $\alpha \cong \tan(\alpha) = -\dot{y}/V_\infty$, letting $\omega = \sqrt{\dfrac{k}{m}},\ \beta = \dfrac{r}{2m\omega},\ \eta = \dfrac{\rho_a d^2 l}{4m}$ and using the non-dimensional quantities $\tau = \omega t,\ \tilde{y} = \dfrac{y}{d},\ U = \dfrac{V_\infty}{\omega d}$, the equation of motion is rewritten as

$$\frac{d^2\tilde{y}}{d\tau^2} + \tilde{y} = -2(\beta + \eta U a_1)\frac{d\tilde{y}}{d\tau} + 2\eta a_2\left(\frac{d\tilde{y}}{d\tau}\right)^2 - \frac{2\eta a_3}{U}\left(\frac{d\tilde{y}}{d\tau}\right)^3 - \frac{2\eta a_5}{U^3}\left(\frac{d\tilde{y}}{d\tau}\right)^5 - \frac{2\eta a_7}{U^5}\left(\frac{d\tilde{y}}{d\tau}\right)^7 \qquad (6)$$

Because the damping and aerodynamic forces are much smaller than the stiffness and inertia forces, the terms in the left hand side of equation (6) are at least one order of magnitude larger than those in the other side. We highlight this by using the small parameter $\varepsilon << 1$:

$$\varepsilon\alpha_1 = -2(\beta + \eta U a_1),\ \varepsilon\alpha_2 = 2\eta a_2,\ \varepsilon\alpha_3 = -2\eta a_3/U,$$
$$\varepsilon\alpha_5 = -2\eta a_5/U^3,\ \varepsilon\alpha_7 = -2\eta a_7/U^5$$

so the equation of motion is transformed into a weakly nonlinear Lienard equation:

$$\frac{d^2\tilde{y}}{d\tau^2} + \tilde{y} = \varepsilon\left(\alpha_1\frac{d\tilde{y}}{d\tau} + \alpha_2\left(\frac{d\tilde{y}}{d\tau}\right)^2 + \alpha_3\left(\frac{d\tilde{y}}{d\tau}\right)^3 + \alpha_5\left(\frac{d\tilde{y}}{d\tau}\right)^5 + \alpha_7\left(\frac{d\tilde{y}}{d\tau}\right)^7\right) \qquad (7)$$

3. Variation iteration method. Basic concepts

Consider a general nonlinear system as follows

$$\mathbf{L}(\mathbf{u}(t)) + \mathbf{N}(\mathbf{u}(t)) = \mathbf{g}(t) \tag{8}$$

where **L** and **N** are linear and nonlinear operators respectively, **g** is a known continuous function and *t* is the time. The basic idea of the method is to construct a correction functional for the system (8) in the form

$$\mathbf{u}_{n+1}(t) = \mathbf{u}_n(t) + \int_0^t \boldsymbol{\lambda}(s)\cdot\left[\mathbf{L}(\mathbf{u}_n) + \mathbf{N}(\tilde{\mathbf{u}}_n) - \mathbf{g}(s)\right]ds \tag{9}$$

where $\boldsymbol{\lambda}$ is a general Lagrange multiplier that can be identified optimally via the variational theory, $\mathbf{u}_n$ is the approximate solution of order *n* and $\tilde{\mathbf{u}}_n$ denotes a restricted variation, i.e $\delta\tilde{\mathbf{u}}_n = 0$ [6].

Thus, for the equation $\frac{d^2u}{dt^2} + \omega^2 u = g(t)$ the Lagrange multiplier results as $\lambda(s) = \frac{1}{\omega}\sin\omega(s-t)$. Having λ determined and using a selective zeroth approximation $\mathbf{u}_0$, several approximations $\mathbf{u}_n, n \geq 1$ can be determined with iterative formula (9). Finally, the solution of problem (8) is given by

$$\mathbf{u}(t) = \lim_{n\to\infty}\mathbf{u}_n(t) \tag{10}$$

4. Application to plunge galloping equation

The starting function $\tilde{y}_0$ is selected considering a small deviation from the case $\varepsilon = 0$

$$\tilde{y}_0(\tau) = a\cos\psi + \varepsilon Y_1 + \varepsilon^2 Y_2 \tag{11}$$

where the amplitude *a* and phase ψ are supposed to be slowly varying in non-dimensional time τ [7, 8]

$$\frac{da}{d\tau} = \varepsilon A_1(a) + \varepsilon^2 A_2(a)\,,\quad \frac{d\psi}{d\tau} = 1 + \varepsilon B_1(a) + \varepsilon^2 B_2(a) \tag{12}$$

By differentiating $\tilde{y}_0$ twice with respect to τ yields

$$\frac{d\tilde{y}_0}{d\tau} = -a\sin\psi + \varepsilon\left(A_1\cos\psi - aB_1\sin\psi + \frac{dY_1}{d\tau}\right) + \varepsilon^2\left(A_2\cos\psi - aB_2\sin\psi + \frac{dY_2}{d\tau}\right)$$

$$\frac{d^2\tilde{y}_0}{d\tau^2} = -a\cos\psi + \varepsilon\left(-2A_1\sin\psi - 2aB_1\cos\psi + \frac{d^2Y_1}{d\tau^2}\right) + \varepsilon^2\left(\left(A_1\frac{dA_1}{da} - aB_1^2 - 2aB_2\right)\cos\psi - (2A_2 + aA_1\frac{dB_1}{da} + 2A_1B_1)\sin\psi + \frac{d^2Y_2}{d\tau^2}\right)$$

The first iterate of (9) is

$$\tilde{y}_1(\tau) = \tilde{y}_0(\tau) + \int_0^\tau \sin(s-\tau)\cdot\left[\frac{d^2\tilde{y}_0}{d\tau^2} + \tilde{y}_0 - \varepsilon\left(\alpha_1\frac{d\tilde{y}_0}{d\tau} + \alpha_2\left(\frac{d\tilde{y}_0}{d\tau}\right)^2 + \alpha_3\left(\frac{d\tilde{y}_0}{d\tau}\right)^3 + \alpha_5\left(\frac{d\tilde{y}_0}{d\tau}\right)^5 + \alpha_7\left(\frac{d\tilde{y}_0}{d\tau}\right)^7\right)\right]ds =$$

$$= \tilde{y}_0(\tau) + \int_0^\tau \sin(s-\tau)\cdot\left\{\varepsilon\left[-2A_1\sin\psi - 2aB_1\cos\psi + d^2Y_1/d\tau^2 + Y_1 - \left(-\alpha_1 a\sin\psi + \alpha_2 a^2\sin^2\psi - \alpha_3 a^3\sin^3\psi - \alpha_5 a^5\sin^5\psi - \alpha_7 a^7\sin^7\psi\right)\right] + \varepsilon^2\left[\left(A_1 dA_1/da - aB_1^2 - 2aB_2\right)\cos\psi - \left(2A_2 + aA_1\cdot dB_1/da + 2A_1B_1\right)\sin\psi + d^2Y_2/d\tau^2 + Y_2 - \left(\alpha_1 - 2\alpha_2 a\sin\psi + 3\alpha_3 a^2\sin^2\psi + 5\alpha_5 a^4\sin^4\psi + 7\alpha_7 a^6\sin^6\psi\right)D\right]\right\}ds \tag{13}$$

with $D = A_1\cos\psi - aB_1\sin\psi + dY_1/d\tau$.

From (12) we conclude that $\psi(\tau) \cong \tau$. It follows that

$$\int_0^\tau \sin(s-\tau)\sin s\,ds = \frac{1}{2}(\tau\cos\tau - \sin\tau)$$

$$\int_0^\tau \sin(s-\tau)\cos s\,ds = \frac{1}{2}\tau\sin\tau\,.$$

Terms like $\tau\sin\tau$ and $\tau\cos\tau$ are called *secular terms* because they increase slowly in time and become important after a long period of time. The plunge galloping is a bounded oscillation, thus we need to avoid the presence of such of terms by cancelling the coefficients of $\sin\psi$ and $\cos\psi$. For the $\mathrm{O}(\varepsilon)$ part of the equation (13) this yields

$$\begin{aligned}&\sin\psi: -2A_1 + \alpha_1 a + \frac{3\alpha_3}{4}a^3 + \frac{5\alpha_5}{8}a^5 + \frac{35\alpha^7}{64}a^7 = 0\\ &\cos\psi: -2aB_1 = 0\end{aligned} \tag{14}$$

From (14) one has

$$A_1 = \frac{\alpha_1}{2}a + \frac{3\alpha_3}{8}a^3 + \frac{5\alpha_5}{16}a^5 + \frac{35\alpha_7}{128}a^7,\ B_1 = 0 \tag{15}$$

In this point we introduce a modification to the classical VIM in that we require $\tilde{y}_1 = \tilde{y}_0$, meaning the approximate solution is obtained after only one iteration. A first step to achieve this is to cancel the rest of the coefficient of $\mathrm{O}(\varepsilon)$ part in (13). It results a linear second-order differential equation in Y_1 having the solution

$$Y_1(\tau) = \frac{\alpha_2 a^2}{2}\left(1 + \frac{\cos 2\psi}{3}\right) - \left(\frac{\alpha_3 a^3}{32} + \frac{5\alpha_5 a^5}{128} + \frac{21\alpha_7 a^7}{512}\right)\sin 3\psi + \left(\frac{\alpha_5 a^5}{384} + \frac{7\alpha_7 a^7}{1536}\right)\sin 5\psi - \frac{35\alpha_7 a^7}{128}\sin 7\psi \tag{16}$$

By avoiding the secular terms in the $\mathrm{O}(\varepsilon^2)$ part of the equation (13) one obtain

$$A_2 = 0, B_2 = -\frac{\alpha_1^2}{8} - \frac{\alpha_2^2}{6}a^2 + \left(\frac{5\alpha_1\alpha_5}{32} - \frac{9\alpha_3^2}{256}\right)a^4 + \left(\frac{35\alpha_1\alpha_7}{128} + \frac{135\alpha_3\alpha_5}{1024}\right)a^6 + \left(\frac{333\alpha_3\alpha_7}{2048} + \frac{875\alpha_5^2}{12288}\right)a^8 + \frac{7910\alpha_5\alpha_7}{49152}a^{10} + \frac{30821\alpha_7^2}{393216}a^{12} \tag{17}$$

Finally, Y_2 is the solution of the second-order differential equation obtained by calling off the remainder of $\mathrm{O}(\varepsilon^2)$ part in (13). In what follows we consider the particular cases studied by Blevins and, respectively Parkinson and Smith.

Case 1: $\alpha_5 = \alpha_7 = 0$

The approximate solution (11) reduces to

$$\tilde{y}(\tau) = a\cos\psi + \varepsilon\left[\frac{\alpha_2 a^2}{2}\left(1 + \frac{\cos 2\psi}{3}\right)\right] - \frac{\varepsilon\alpha_3 a^3}{32}\sin 3\psi \tag{18}$$

with *a* and ψ given by

$$\begin{aligned}\frac{da}{d\tau} &= \frac{\varepsilon\alpha_1}{2}a + \frac{3\varepsilon\alpha_3}{8}a^3,\\ \frac{d\psi}{d\tau} &= 1 - \frac{(\varepsilon\alpha_1)^2}{8} - \frac{(\varepsilon\alpha_2)^2}{6}a^2 - \frac{9(\varepsilon\alpha_3)^2}{256}a^4\end{aligned} \tag{19}$$

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For the steady-state solution, $da/d\tau = 0$. If $\beta + \eta U a_1 > 0$, one has a single solution, corresponding to the equilibrium configuration (a = 0). At $U_b = -\dfrac{\beta}{\eta a_1}$ a Hopf bifurcation appears and the system performs a plunge oscillation with amplitude

$$a = \sqrt{-\frac{4\varepsilon\alpha_1}{3\varepsilon\alpha_3}} = \sqrt{-\frac{4(\beta + \eta U a_1)}{3\eta a_3}} \qquad (20)$$

Case 2: $\alpha_2 = 0$

The oscillation is described by the law

$$\tilde{y}(\tau) = a\cos\psi - \varepsilon\left(\frac{\alpha_3 a^3}{32} + \frac{5\alpha_5 a^5}{128} + \frac{21\alpha_7 a^7}{512}\right)\sin 3\psi + $$
$$+\varepsilon\left(\frac{\alpha_5 a^5}{384} + \frac{7\alpha_7 a^7}{1536}\right)\sin 5\psi - \frac{35\,\varepsilon\alpha_7 a^7}{128}\sin 7\psi \qquad (21)$$

where the amplitude a verifies the differential equation

$$\frac{da}{d\tau} = \frac{\varepsilon\alpha_1}{2}a + \frac{3\varepsilon\alpha_3}{8}a^3 + \frac{5\varepsilon\alpha_5}{16}a^5 + \frac{35\varepsilon\alpha_7}{128}a^7 \qquad (22)$$

The algebraic equation of non-zero steady amplitudes is a cubic equation for a^2:

$$(\beta + \eta U a_1) + \frac{3\eta a_3}{4U}a^2 + \frac{5\eta a_5}{8U^3}a^4 + \frac{35\eta a_7}{64U^5}a^6 = 0 \qquad (23)$$

The number of real roots (one or three) depends on the coefficients [5].

5. Numerical results

In this section we have carried out a number of numerical simulations for verifying the accuracy of the approximate solution. We fixed $\beta = 0.014, \eta = 0.0005$ and chose U as a parameter.

Case 1: $\alpha_5 = \alpha_7 = 0$

According to Blevins, $a_1 = -2.7, a_2 = -1.8$ and $a_3 = 31$ (see Fig. 2 a). The bifurcation point is $U_b = 10.37$, so we expect the system comes to rest if $U < U_b$ or oscillates in accordance with the formulas (18 -20) on the contrary.

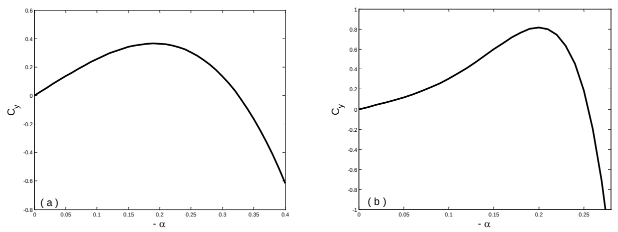

Fig. 2. *The lateral force coefficient C_y as a function of angle $-\alpha$ a) Case 1; b) Case 2.*

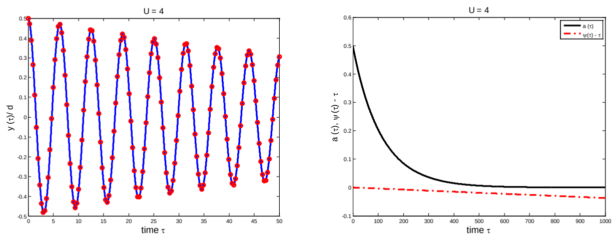

Fig. 3. *Left: The comparison between the time series solutions $y(\tau)/d$ obtained with exact and approximate equations (6) and (18) for $U = 4$. The red dots are for solution (18); Right: Evolution in time of amplitude a and of difference between phase ψ and time τ for $U = 4$.*

Indeed, for $U = 4 < U_b$ an initial displacement $\tilde{y}(0) = 0.5$ is slowly attenuated by the damping mechanism, as illustrated in Fig. 3. There exist an excellent agreement between the numerical solution (continuous line), obtained by direct integration of eq. (6), and its approximate counterpart (18) (red small circles). The angle ψ increases almost linearly with τ (the relative errors between them does not exceed 0.01%).

By increasing the speed at $U = 20 > U_b$ the structure evolves into a plunge oscillation with the steady amplitude a = 4.7292. Fig. 4 shows, on the one hand, extracts of the transitional and stationary phases and, on the other hand, a high accuracy of the approximation (18).

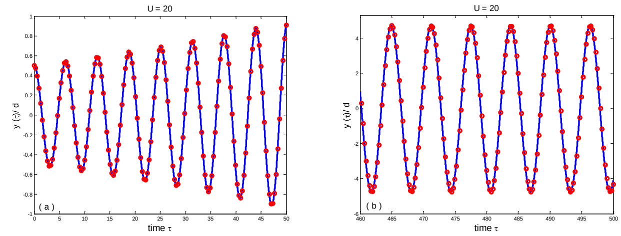

Fig. 4. *The comparison between the time series solutions $y(\tau)/d$ obtained with exact and approximate equations (6) and (18) for $U = 20$. Red dots are for solution (18): (a) transitional phase; (b) stationary phase*

The necessary time to reach the steady-state behavior has a maximum near the bifurcation point U_b. In Fig. 5 are reported as a function of speed U the amplitudes of oscillation determined from equations (20), (18) and (6). If the plot is realized after $\Delta\tau = 300$ units of time, then an important difference between solutions given by (20), on one hand, and (18) and (6), on the other hand, is easily observed. The system did not have enough time to pass over the transient. An interval of $\Delta\tau = 2000$ units of time seems to be a good option for ensure the overlap of the three solutions.

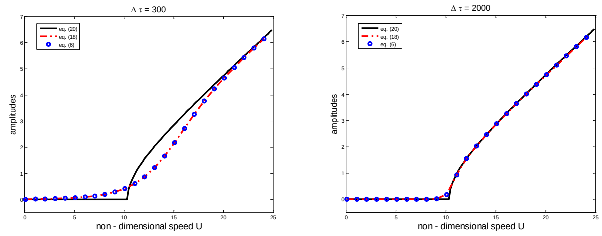

Fig. 5. *The galloping amplitude as a function of speed Left: $\Delta\tau = 300$; Right: $\Delta\tau = 2000$*

Case 2: $\alpha_2 = 0$

We settled out the aerodynamic coefficients at the values used by Parkinson and Smith, namely $a_1 = -2.69, a_3 = 168, a_5 = -6270$ and $a_7 = 59900$. The lateral force coefficient C_y curve is plotted in Fig. 2 b. For small speeds U any initial disturbance is slowly neutralized by damping (see Fig. 6, left panel). The influence of $O(\varepsilon)$ terms in (21) is of little impact on the resulting motion.

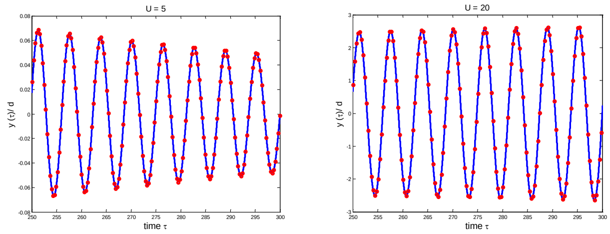

Fig. 6. *The comparison between the time series solutions $y(\tau)/d$ obtained with exact and approximate equations (6) and (21) for $U = 5$ (left) and $U = 20$ (right). The red dots are for solution (21).*

If U is further increased, at $U_b = 10.41$ a Hopf bifurcation changes the system's behavior, which performs a plunge oscillation with an amplitude proportionally to the speed. The right panel of Fig. 6 presents a fragment of the time series solutions for U = 20, obtained with (6) and (21) and where the $O(\varepsilon)$ terms were not included. The two solutions agree extremely well, excepting a small interval of time before the transition to be completed (see Fig. 7).

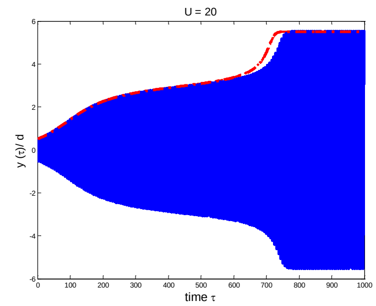

Fig. 7. *The time evolution of the oscillatory process for U = 20. The red interrupted curve stands for the solution of (22).*

For large amplitudes of oscillation, terms like $\alpha_5 a^5$ or $\alpha_7 a^7$ in (21) become important and distort in an unacceptable manner the sinusoidal shape of curve $\tilde{y}(\tau)$, as shown in Fig. 8.

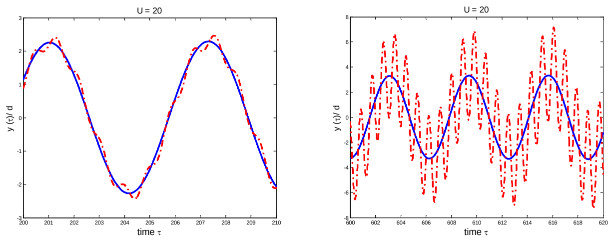

Fig. 8. *The distortion of numerical solution (blue continuous curve) produced by terms containing $\alpha_n a^n, n = 3,5,7$ in (21).*

The algebraic equation (23) of non-zero steady amplitudes may have one or three positive roots, depending on the speed *U*.

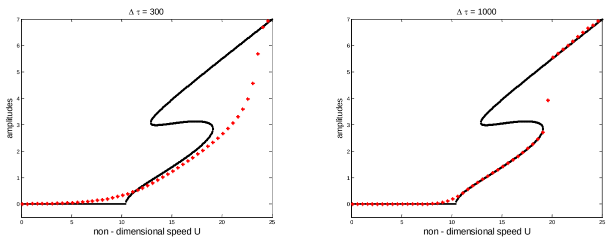

Fig. 9. *The galloping amplitude as a function of speed. Red dots stand for numerical solution while the black curve shows the roots of (23). The initial conditions $(\tilde{y}(0), d\tilde{y}(0)/d\tau) = (0.5, 0)$ were used.*
Left: $\Delta\tau = 300$; Right: $\Delta\tau = 1000$

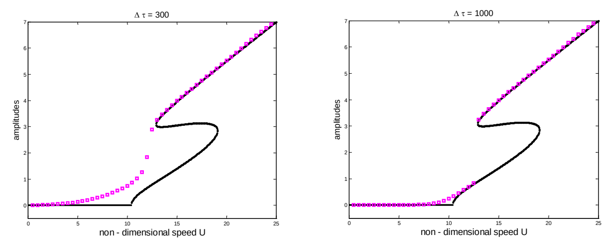

Fig. 10. *The galloping amplitude as a function of speed. Magenta squares stand for numerical solution while the black curve shows the roots of (23). The initial conditions $(\tilde{y}(0), d\tilde{y}(0)/d\tau) = (4.5, 0)$ were used.*
Left: $\Delta\tau = 300$; Right: $\Delta\tau = 1000$

This leads to a region of oscillation hysteresis, between $U_{\min} = 12.9$ and $U_{\max} = 19.1$. For $U \in (U_{\min}, U_{\max})$, the intermediate root corresponds to an unstable solution and the amplitude of oscillation is equal to one of the remaining roots, function on the initial conditions. Again, a relative long period of time is required for reaching the steady-state solution (see Figs. 9 and 10).

6. Conclusions

In the paper, we applied a modified variational iteration method to the plunge galloping problem for deriving a system of two differential equations describing the rate of change of the amplitude and frequency of zeroth order solution. Two possible behaviors have been deduced and they can be either an equilibrium state or a periodic plunge oscillation. Using a typical iced covered cross-section of an electrical transmission line and two expressions for the vertical aerodynamic force coefficient, we founded an excellent agreement between the results provided by numerical integration of the equation of motion and zeroth order approximate solution obtained with variational iteration method. There is no need for higher order approximations.

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