ON APPROXIMATING THE PERIODIC SOLUTIONS OF CAPSIZE EQUATION

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Abstract: *The motion of a ship in long beam seas could be described by a second-order non-linear differential equation, having the roll angle as variable and depending on four parameters. With the direct forcing amplitude as bifurcation parameter, the dynamical system exhibits either periodic or chaotic behaviour, the route to chaos being realized by a period doubling sequence of periodic motions. Some accepted indicators, like bifurcation diagrams, phase planes and Poincare sections have been computed and they confirm the transition from order to chaos. In the main part of the paper, the harmonic balance method is used to obtain approximate solutions for the periodic motions and to predict the period doubling bifurcations by a stability analysis.*

Keywords: SHIP CAPSIZE, DIRECT AND PARAMETRIC EXCITATION, ORDER AND CHAOS, PERIODIC SOLUTIONS

1. Introduction

Capsizing is the most disastrous situation in which can find a sailing ship, since it leads to heavy losses of human lives and ship. It primarily affects the fishing boats and small vessels, but in certain circumstances large ships may suffer too, although they satisfy all the existing rules concerning the risk of capsizing [1]. This is why the stability against capsizing is a fundamental requirement when designing a ship. Unfortunately, despite an extensive experimental and theoretical research during the last decades, this undesirable complex phenomenon is not fully understood yet.

The efforts of researchers have been focused on several directions. First, it was found that the majority of capsizing events occur in astern seas, as a result of pure loss of stability, parametric resonance, broaching, cargo shift, water on deck and wind [2]. Second, mathematical models, following an increasing sophistication and capable of predicting well enough the ship dynamics for some given environment conditions, have been proposed. Most of these approaches consider only the ship's rolling motion, but models with two, three and even six degrees-of-freedom have been developed. A generally accepted model has not yet been established [3, 4]. Finally, some innovative concepts as transient safe basins, transient capsize diagram, index of capsized resistance, and the recent theoretical developments in the dynamical non-linear systems seem to lead to an improved understanding of the ship capsize process [5, 6].

According to Cardo et al [7], it has become habitual to study the capsize dynamics using a one degree-of-freedom nonlinear oscillator taking the form $\ddot{\theta} + g\left(\dot{\theta}\right) + GZ(\theta,t) = f(t)$, with θ the roll angle, $g\left(\dot{\theta}\right)$ the damping function, $GZ(\theta,t)$ a non-linear function which encapsulates the restoring moment, and $f(t)$ the external forcing function depending on time.

In the present work we shall restrict our attention to the particular case of Thompson's equation, which models in a simplified fashion the uncoupled roll motion of a ship in periodic beam seas, with both direct and parametric excitation, the latter multiplying the conventional restoring function [8]. The remainder of our paper proceeds as follows. In Section 2 the archetypal model equation of Thompson et al. is presented in the simplest setting possible. In the following two sections, the approximate solutions for the periodic orbits, given by fast Fourier transform and harmonic balance method, are obtained. We close with a short summary and conclusions in Section 5.

2. Capsize equation

In the paper, the following equation, derived by Thompson et al., is further investigated numerically and analytically with a view to prove the period doubling sequence of periodic motions leading to chaos, and finally to capsize:

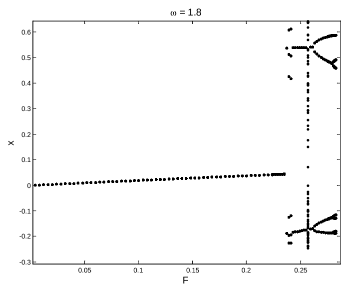
(1) $$\ddot{x} + \beta\dot{x} + \left(x - x^2\right)\left(1 + G\cos\omega t\right) = F\sin\omega t$$

where

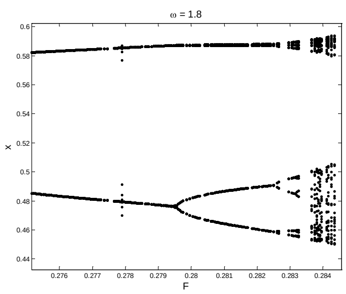
(2) $$x = \frac{\theta}{\theta_V},\ \omega = \frac{\omega_f}{\omega_n},\ F = -\frac{Ak\omega^2}{\theta_V},\ G = -Ak$$

Here, θ represents the roll angle, θ_V the angle of vanishing stability, β the non-dimensional damping coefficient, ω_f the wave frequency, ω_n the natural frequency of the boat, A the wave height, and k the wave number. The forcing amplitudes F and G stand for direct and parametric excitation, respectively. Finally, a dot denotes differentiation with respect to time. The details about Eq. 1 can be found in [8], where some preliminary numerical studies for $\omega = 0.85$ and $\beta = 0.1$ are presented.

As a continuation of their work, in the present paper we address a different area of frequencies, namely we set $\omega = 1.8$. Other constant values used are $\beta = 0.1$, and $G = 5F$. With the direct forcing amplitude F as bifurcation parameter, it was found that the analyzed system exhibits either periodic or chaotic behavior, the route to chaos being realized by a doubling sequence of periodic motions. A sense of the rich dynamical behavior which system (1) displays can be gleaned from the bifurcation diagram $F - x$ in Fig.1.



(a)



(b)

Fig. 1 *a) Bifurcation diagram F - x for* $\beta = 0.1, \omega = 1.8, G = 5F$;
b) The upper – right part of (a) is zoomed for details.

To construct this diagram, we started from equilibrium conditions ($x(0)=\dot{x}(0)=0$) and plotted x at every one forcing cycle with the same phase angle. The first 200 cycles were discarded to avoid transients.

3. Approximate solutions for periodic orbits with Fast Fourier Transform

Fig. 1 shows that for relatively small forcing amplitude F the system executes oscillations with period $T=2\pi/\omega$. As F is gradually increased the period –*T* motion bifurcates into a period –*2T* motion. For further increase of forcing period *4T, 8T, 16T*, and so on, motions are obtained. When F outruns 0.286, the response becomes chaotic. The periodic motions could be well approximated by a finite series of the form

$$x(t)=\sum_{i=0}^{n}\left(a_i\cos\omega t/k+b_i\sin\omega t/k\right) \qquad (3)$$

with coefficients $a_i, b_i, i=\overline{0,n}$, real numbers given by *Fast Fourier Transform* (for short *FFT*) and k stands for *kT* period. The phase plane plots, and the Fourier spectra for the cosine and sine components (black and red vertical lines, respectively) when $F\in\{0.1, 0.25, 0.27, 0.282\}$ are illustrated in Figs. 2 to 5. The results obtained by FFT method, which include the first 12 harmonics and sub-harmonics, are also presented in the phase planes and are indicated by red asterisks. The blue dots stand for Poincare sections. It is obvious that the agreement between the solution obtained by the FFT method and the numerical integration is excellent, in what when one is superimposed over the other they are practically indistinguishable.

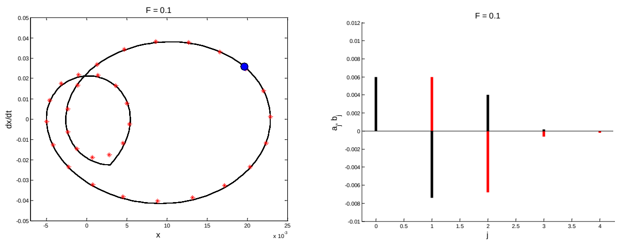


Fig. 2 *Phase plane plots and the Fourier spectra for capsize equation (1) with F = 0.1.*

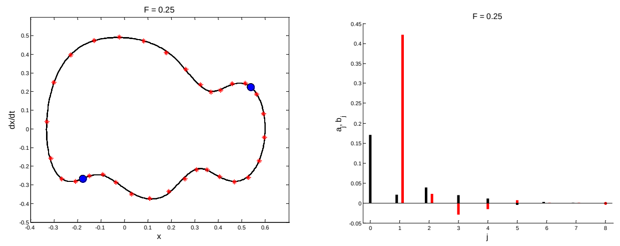


Fig. 3 *Phase plane plots and the Fourier spectra for capsize equation (1) with F = 0.25.*

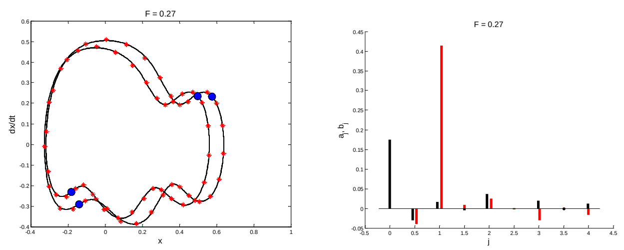


Fig. 4 *Phase plane plots and the Fourier spectra for capsize equation (1) with F = 0.27.*

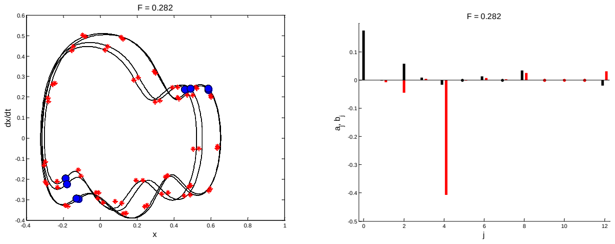


Fig. 5 *Phase plane plots and the Fourier spectra for capsize equation (1) with F = 0.282.*

4. Approximate solutions for periodic orbits with harmonic balance method

From the dynamical phase diagram depicted in Fig. 1, it is clear that the period bifurcation phenomenon could be regarded as a precursor of the chaotic behavior and, finally, of ship capsizing. To estimate approximately the bifurcation point from period *T* – orbit to period *2T* – orbit and the significant jump of the roll amplitudes, in this section a harmonic balance analysis is performed [9].

According to experience gained in working with FFT, we assume a period *T* solution of the form (3), with n = 2 and k = 1. Substituting it in the capsize equation (1) and balancing the free terms and the coefficients of $\cos\omega t, \sin\omega t, \cos 2\omega t$ and $\sin 2\omega t$, the following strong non-linear system of five equations with a_0, a_1, a_2, b_1 and b_2 as unknowns is obtained:

$$a_0(1-a_0)-0.5\left(a_1^2+b_1^2+a_2^2+b_2^2\right)+a_1G(0.5-a_0)+ \\ +0.5a_2G-0.5G(a_1a_2+b_1b_2)=0 \qquad (4)$$

$$a_0G(1-a_0-a_2)-0.5G\left(2a_1^2+a_2^2+b_2^2\right)+a_1\left(1-\omega^2-2a_0\right)+ \\ +b_1\omega\beta-(a_1a_2+b_1b_2)=0 \qquad (5)$$

$$-a_1\omega\beta+b_1\left(1-\omega^2-2a_0\right)-(a_1b_1+a_0b_2)G+b_1a_2-b_2a_1=F \qquad (6)$$

$$a_1G(0.5-a_0)+a_2\left(1-4\omega^2-2a_0\right)+2b_2\omega\beta- \\ -a_1a_2G-0.5\left(a_1^2-b_1^2\right)=0 \qquad (7)$$

$$b_1G(0.5-a_0)+b_2\left(1-4\omega^2-2a_0\right)-2a_2\omega\beta- \\ -a_1b_1+b_1b_2G=0 \qquad (8)$$

This system has been solved for different forcing amplitudes F, equally spaced between 0 and 0.22, using Newton-Raphson method and the findings are displayed in Fig. 6.

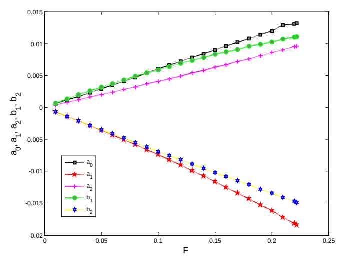


Fig. 6 *Amplitudes* a_0, a_1, a_2, b_1 *and* b_2 *for different forcing amplitudes*

For F = 0.1, we found $a_0=0.0062, a_1=-0.0077, a_2=0.0041$, $b_1=0.006$, and $b_2=-0.0067$. The associated time history and the phase plane (the asterisks) are shown in Fig. 7, along with the solution obtained by numerical integration (the continuous black curve). It could be seen that the agreement between the two solutions is good, with some notable differences at peaks and troughs of time history plot.

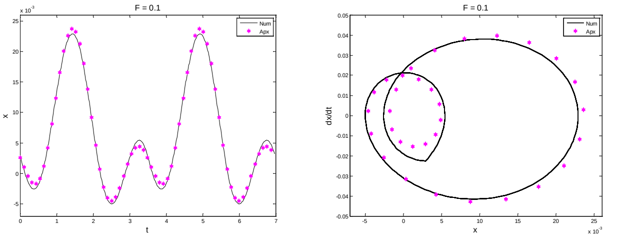


Fig. 7 *Time history and phase plane for capsize equation with F = 0.1.*

Magenta asterisks stand for approximate solution given by harmonic balance method, while black curves represent numerical solution.

Continuing to grow the forcing amplitude F above 0.22 will come a time when the period – T solution becomes unstable and a period *2T*- orbit will replace it. To investigate this bifurcation one considers a perturbed solution of the form $\overline{x}(t)=x(t)+\delta x(t)$. Replacing the perturbed solution into (1) and noting that *x(t)* verifies the same equation, we get the following linearized variational equation

(9) $$\ddot{\delta x}+\beta\,\dot{\delta x}+(1-2x)(1+G\cos\omega t)\delta x=0$$

Fig. 3 shows us that the perturbation δx must be chosen as

(10) $$\delta x=a_{1/2}\cos\frac{\omega t}{2}+b_{1/2}\sin\frac{\omega t}{2}+a_{3/2}\cos\frac{3\omega t}{2}+b_{3/2}\sin\frac{3\omega t}{2}$$

From (10) and (9), by equating like harmonic terms, a homogeneous linear algebraic system is obtained. In matrix form, it is written as

(11) $$[M]\cdot\{a_{1/2}\; b_{1/2}\; a_{3/2}\; b_{3/2}\}^T=\{0\quad 0\quad 0\quad 0\}^T$$

where the elements of matrix [*M*], which are functions of a_0, a_1, a_2, b_1 and b_2, are given in Appendix.

The period – 1 solution becomes unstable and bifurcates into period – 2 solution when the determinant of matrix M changes its sign from negative to positive [9]. For the analyzed case, this happens when $F\cong 0.22$, which is in good agreement with value $F\cong 0.23$ given by numerical integration (see Fig. 8).

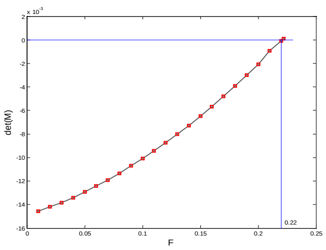


Fig. 8 *Determinant of matrix M vs. forcing amplitude F*

Further, for a brief interval of forcing amplitudes' values the system (1) undergoes a transition from a period 1 to a period 2 motion, this stage being characterized by high-order period motions or even chaos (see Fig. 9).

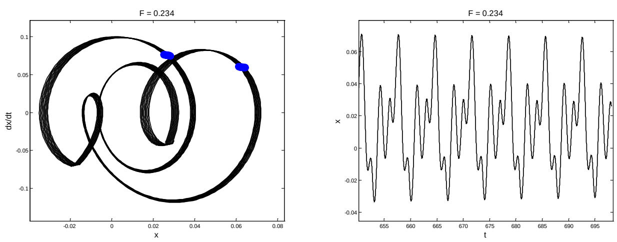


Fig. 9 *Time history and phase plane for capsize equation with F = 0.234.*

Starting with $F\cong 0.232$, the equation (1) has a solution of the form $\overline{x}(t)$, which includes the perturbation δx given by (10). Replacing it into (1) and balancing the harmonic terms, the following algebraic non-linear system of nine equations is obtained:

(12) $$a_0(1-a_0)-0.5\left(a_{1/2}^2+b_{1/2}^2+a_1^2+b_1^2+a_{3/2}^2+b_{3/2}^2+a_2^2+b_2^2\right)+$$
$$+G\left[0.5(a_1-a_{1/2}a_{3/2}-b_{1/2}b_{3/2}-a_1a_2)+\left(b_{1/2}^2-a_{1/2}^2\right)/4-a_0a_1\right]=0$$

(13) $$a_{1/2}\left(1-0.25\omega^2-2a_0\right)+0.5\beta\omega b_{1/2}-(a_{1/2}a_1+b_{1/2}b_1+$$
$$+a_1a_{3/2}+a_{3/2}a_2+b_{3/2}b_2)+G[0.5(a_{1/2}+a_{3/2}-a_{1/2}a_2-b_{1/2}b_2-$$
$$-a_1a_{3/2}-a_{3/2}a_2-b_{3/2}b_2)-(a_0a_{1/2}+a_0a_{3/2}+a_{1/2}a_1)]=0$$

(14) $$-0.5\beta\omega a_{1/2}+b_{1/2}\left(1-0.25\omega^2-2a_0\right)-(a_{1/2}b_1-b_{1/2}a_1+$$
$$+a_1b_{3/2}+a_{3/2}b_2-b_{3/2}a_2)+G[0.5(-b_{1/2}+b_{3/2}-a_{1/2}b_2+b_{1/2}a_2+$$
$$+a_1b_{3/2}+a_{3/2}b_2-b_{3/2}a_2)+(a_0b_{1/2}-a_0b_{3/2}-a_{1/2}b_2)]=0$$

(15) $$a_1\left(1-\omega^2-2a_0\right)+\beta\omega b_1+0.5\left(b_{1/2}^2-a_{1/2}^2\right)-(a_{1/2}a_{3/2}+$$
$$+b_{1/2}b_{3/2}+a_1a_2)+G\left[a_0(1-a_0)+0.5\left(a_2-a_{1/2}^2-b_{1/2}^2-a_{3/2}^2-b_{3/2}^2-\right.\right.$$
$$\left.\left.-a_2^2-b_2^2-a_{1/2}a_{3/2}+b_{1/2}b_{3/2}\right)-0.25\left(3a_1^2+b_1^2\right)-a_0a_2\right]=0$$

(16) $$-\beta\omega a_1+b_1\left(1-\omega^2-2a_0\right)-(a_{1/2}b_{1/2}+a_{1/2}b_{3/2}-b_{1/2}a_{3/2})+$$
$$+G[0.5(b_2-a_{1/2}b_{3/2}-b_{1/2}a_{3/2}-a_1b_1)-a_0b_2]=F$$

(17) $$a_{3/2}\left(1-2.25\omega^2-2a_0\right)+1.5\beta\omega b_{3/2}-(a_{1/2}a_1+a_{1/2}a_2-$$
$$-b_{1/2}b_1+b_{1/2}b_2)+G[0.5(a_{1/2}-a_{1/2}a_1-a_{1/2}a_2-b_{1/2}b_1+b_{1/2}b_2-$$
$$-a_{3/2}a_2-b_{3/2}b_2)-(a_0a_{1/2}+a_1a_{3/2})]=0$$

(18) $$-1.5\beta\omega a_{3/2}+b_{3/2}\left(1-2.25\omega^2-2a_0\right)-(a_{1/2}b_1+a_{1/2}b_2+$$
$$+b_{1/2}a_1-b_{1/2}a_2)+G[0.5(b_{1/2}-a_{1/2}b_1-a_{1/2}b_2+b_{1/2}a_1-b_{1/2}a_2-$$
$$-a_{3/2}b_2+b_{3/2}a_2)-(a_0b_{1/2}+a_1b_{3/2})]=0$$

(19) $$a_2\left(1-4\omega^2-2a_0\right)+2\beta\omega b_2+0.5\left(b_1^2-a_1^2\right)-(a_{1/2}a_{3/2}+$$
$$-b_{1/2}b_{3/2})+G[0.5(a_1-a_{1/2}a_{3/2}-b_{1/2}b_{3/2})-(a_0a_1+a_1a_2)+$$
$$+0.25\left(b_{1/2}^2-a_{1/2}^2+b_{3/2}^2-a_{3/2}^2\right)]=0$$

(20) $$-2\beta\omega a_2+b_2\left(1-4\omega^2-2a_0\right)-(a_{1/2}b_{3/2}+b_{1/2}a_{3/2}+a_1b_1)+$$
$$+G[0.5(b_1-a_{1/2}b_{1/2}+b_{1/2}a_{3/2}-a_{3/2}b_{3/2})-(a_0b_1+a_1b_2)]=0$$

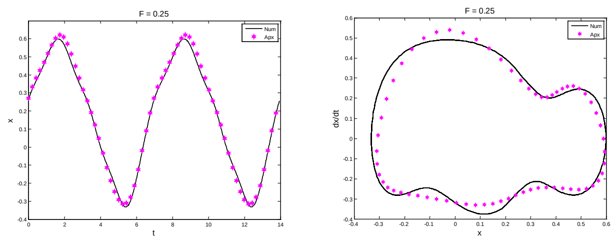


Fig. 10 *Time history and phase plane for capsize equation with F = 0.25.*

Magenta asterisks stand for approximate solution given by harmonic balance method, while black curves represent numerical solution.

The Newton –Raphson procedure did not work for the system (12-20) due to the important difference between a_0 and $b_{1/2}$ on one hand, and the other unknowns.

However, neglecting the small terms like $a_{1/2}a_2$, we were able to estimate the solution of system (12 – 20). The results for $F = 0.25$ are shown in Fig. 10. Again, the agreement between the harmonic balance solution and the numerical one is good enough, the notable differences being associated to the first derivative, dx/dt. The period – $2T$ motion loses quickly its stability (around $F \cong 0.26$) and, after a new transition stage, turn into a period – $4T$ motion. The mathematics involved by harmonic balance method becomes from now on quite cumbersome.

5. Conclusions

In the paper, the ship capsize equation provided by Thompson and co-workers was analytically and numerically investigated for a range of frequencies far from resonance conditions. Excepting the fact that capsizing is moving into an area of higher values of forcing amplitudes, the system's behaviour remains unchanged, i.e. the chaos is shown to occur through a cascade of period doubling bifurcations as the forcing amplitude is increased.

The *Fast Fourier Transform* has been used to obtain approximate solutions for period – T, $2T$, $4T$ and $8T$ motions, and the results proved to be excellent when compared with the numerical ones.

The *Harmonic Balance Technique*, with appropriate harmonic terms in the assumed solution, allowed us to approximate period T and $2T$ motions and to estimate the bifurcation between them. Because of the mathematics involved in approximating the higher-order period motions, the above-mentioned method is inadequate to study the next period-doubling bifurcations leading to chaos.

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APPENDIX

The elements $m_{ij}, 1 \le i, j \le 4$ of matrix M in (11) are defined as follows:

$$P_1 = 1 - 2a_0 - a_1 G - 0.25\omega^2,\ P_2 = 0.5(1 - 2a_0 - a_2)G - a_1,$$

$$P_3 = -b_1 - 0.5 b_2 G,\ P_4 = 0.5\beta\omega, P_5 = a_2 + 0.5 a_1 G,$$

$$P_6 = -b_2 - 0.5 b_1 G,\ P_7 = 1 - 2a_0 - a_1 G - 2.25\omega^2, P_8 = 0.5 a_1 G,$$

$$P_9 = 1.5\beta\omega, P_{10} = -0.5 b_2 G, m_{11} = P_1 + P_2, m_{12} = P_3 + P_4,$$

$$m_{13} = P_2 - P_5, m_{14} = P_6 + P_3, m_{21} = P_3 - P_4, m_{22} = P_1 - P_2,$$

$$m_{23} = P_6 - P_3, m_{24} = P_2 + P_5, m_{31} = P_2 - P_5, m_{32} = P_6 - P_3,$$

$$m_{33} = P_7 - P_8, m_{34} = P_9 + P_{10}, m_{41} = P_3 + P_6, m_{42} = P_5 + P_2,$$

$$m_{43} = P_{10} - P_9, m_{44} = P_7 + P_8.$$